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Collisionless Shocks and Solitary Waves

CATHLEEN S. MORAWETZ

November 30, 1964

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Collisionless Shocks and Solitary Waves

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Recently there has been a revival of interest in the collision-free magneto-hydrodynamic shock. This interest centers around the apparent detached shock which buffers the earth and its magnetic field from the solar wind. Observations by various satellites indicate that a phenomenon occurs as in Figure 1. Around the earth there is a large vacuum field. A stream of plasma, the solar wind, has to pass around this vacuum field which acts roughly like a body in a supersonic stream. The solar wind is slowed down by crossing a shock and then slips around the vacuum field.

The Lundquist equations and the theory of the detached shock corroborate this picture qualitatively.[1,2] But certain details are contradictory. Namely the plasma layer between the shock and the vacuum field or magnetosphere appears to have a turbulent or at any rate highly irregular magnetic field and plasma velocity. Secondly we know theoretically that there are virtually no collisions, hence no viscosity, and thus the classical theory to

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1. John R. Spreiter and William Pritchard Jones, Journal of Geophysical Research 68, 12 (1963).
2. C.P. Sonett, Bull. National Academy of Sci., 84 (1964).

justify a sharp shock is gone. Finally ahead of the shock edge the orientation of the field has an oscillation.

The early interest in collision-free shocks was mainly connected to the problem of heating rarefied plasmas.[3] In ordinary gases, a natural method for friction is to compress a gas quickly and non-adiabatically by driving a sharp shock through it. Can the same effect be achieved in a plasma without standard dissipative forces?

To look at these problems theoretically we want to consider an idealized mathematical model. The plasma is composed of two kinds of particles, electrons and ions. We suppose that there are no collisions and that the distribution functions $f_{\pm}(\vec{u}, \vec{x}, t)$ for electrons and ions satisfy the Vlasov equations. The electric and magnetic fields $(\vec{E},\vec{B})$ are governed by Maxwell's laws where the current J and charge density q are found as integrals of the distribution functions $f_{\pm}$:

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3. C.S. Gardner, H. Goertzel, H. Grad, C.S. Morawetz, M.H. Rose, and H. Rubin, "Hydromagnetic Shock Waves in High-temperature Plasma", in Proc. of the Second United Nations International Conference on the Peaceful Uses of Atomic Energy, Vol. 31, United Nations, Geneva (1958), 230-237.

$$(1) \qquad \frac{\partial}{\partial t} f_{\pm} + u \cdot \nabla_x f_{\pm} + \left(\frac{\pm e}{m_{\pm}}\right)[E + u \times B] \cdot \nabla_u f_{\pm} = 0$$

$$(2) \qquad \nabla \cdot B = 0 \qquad\qquad \text{curl } B = \mu J$$

$$K \nabla \cdot E = q \qquad\qquad \text{curl } E + \frac{\partial}{\partial t} B = 0$$

$$(3) \qquad J = e \int (f_+ - f_-) u |du|$$

$$q = e \int (f_+ - f_-) |du| \ .$$

Here e is charge per particle, $m_{\pm}$ is mass, μ is permeability and K is small in our units of length.

We consider that all quantities depend on only one space variable[4] x and time t. Furthermore we assume charge neutrality, $(K = 0)$, instead of Poisson's law:

$$(4) \qquad \frac{\partial}{\partial t} f_+ + u_1 \frac{\partial f_+}{\partial x} + \frac{e}{m} (E + u \times B) \cdot \nabla_u f_+ = 0$$

$$(5) \qquad \frac{\partial}{\partial t} f_- + u_1 \frac{\partial f_-}{\partial x} - \frac{e}{\varepsilon^2 m} (E + u \times B) \cdot \nabla_u f_- = 0$$

$$(6) \qquad B_1 = \text{constant}, \qquad \frac{\partial B_3}{\partial x} = -\mu J_2 \ , \qquad \frac{\partial B_2}{\partial x} = \mu J_3 \ ,$$

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4. With three space variables we would have to neglect second moments involving magnetic fields and at most one flow quantity. For high pressure plasmas this is more reasonable.

(7) $q = 0$, $\quad \frac{\partial B_2}{\partial t} - \frac{\partial E_3}{\partial x} = 0$, $\quad \frac{\partial B_3}{\partial t} + \frac{\partial E_2}{\partial x} = 0$.

Here $\varepsilon^2 = m_-/m_+$.

The first standard step is to take moments. We denote by $\langle Q \rangle$, the integral $\int Q|du|$. The equations for the conservation of total mass, total momentum and total energy are then:

$$(8) \quad \frac{\partial}{\partial t} \left\langle f_x + \varepsilon^2 f_- \right\rangle + \frac{\partial}{\partial x} \left\langle u_1(f_+ + \varepsilon^2 f_-) \right\rangle = 0$$

$$(9) \quad \frac{\partial}{\partial t} \left\langle u(f_+ + \varepsilon^2 f_-) \right\rangle + \frac{\partial}{\partial x} \left\langle u_1 u(f_+ + \varepsilon^2 f_-) \right\rangle + \frac{1}{m} B \times \left\langle u(f_+ - f_-) \right\rangle = 0$$

$$(10) \quad \frac{\partial}{\partial t} \left\langle |u|^2 (f_+ + \varepsilon^2 f_-) \right\rangle + \frac{\partial}{\partial x} \left\langle u_1 |u|^2 (f_+ + \varepsilon^2 f_-) \right\rangle - \frac{2}{m} E \cdot \left\langle u(f_+ - f_-) \right\rangle = 0.$$

On the other hand from the ion equation alone we shall use only conservation of mass and momentum in the y,z directions:

$$(11) \quad \frac{\partial}{\partial t} \left\langle f_+ \right\rangle + \frac{\partial}{\partial x} \left\langle u_1 f_+ \right\rangle = 0$$

$$(12) \quad \frac{\partial}{\partial t} \left\langle u f_+ \right\rangle + \frac{\partial}{\partial x} \left\langle u_1 u f_+ \right\rangle - \frac{e}{m_+} \left\langle (E + u \times B) f_+ \right\rangle = 0 \quad (y,z \text{ components})$$

Next we integrate these equations and Maxwell's equations with respect to t, average over a large time T and assume that there are no gross changes with respect to time, i.e.,

$\frac{1}{T}\int_{-T}^{+T} \frac{\partial}{\partial t} F \, dt = 0$. Such states will be called quasi-steady. We are left then with steady state equations for the time average moments and the time average fields. With some computation and integrating we obtain the total conservation equations:

(13) Mass $\left\langle u(f_{+}+\varepsilon^2 f_{-})\right\rangle = \rho u_1 = \text{const.}$

(14) Mom. $$\left\langle u_1 u(f_{+}+\varepsilon^2 f_{-})\right\rangle + \frac{1}{m_+}\begin{Bmatrix} \frac{1}{2}(B_2^2+B_3^2) \\ B_1 B_2 \\ B_1 B_3 \end{Bmatrix} = \text{const.}$$

(15) Energy $$\left\langle |u^2| u_1 (f_{+}+\varepsilon^2 f_{-})\right\rangle + \frac{2}{m_+}(E_2 B_3 - E_3 B_2) = \text{const.}$$

(16) $$E_2 = E_3 = \text{const.} \qquad B_1 = \text{const.}$$

where all quantities are time averages.

From the ion equation (4) by taking moments we obtain similarly $\frac{\partial}{\partial x}\left\langle u_1 f_+\right\rangle = 0$ or

(17) $$\left\langle u_1 f_+\right\rangle = \text{constant}$$

(18) $$\frac{\partial}{\partial x}\left\langle u_1 u f_+\right\rangle - \frac{e}{m_+}\left\langle (E + u \quad B)\, f_+\right\rangle = 0.$$

We finally average the last equation by considering

$\frac{m_+}{ex}\int_{-x}^{+x} dx$ operating on it. Neglecting terms of order ε^2 and using the conservation laws (16) and (17) one finally obtains $E + \bar{u} \times \bar{B} = 0$ in y,z directions.

Here the bar denotes the average over both space and time, i.e.:

$$\bar{u} = \frac{\int uf|du|dxdt}{\int f|du|dxdt} .$$

We also take space averages on the conservation laws and we obtain finally the transition conditions for quasi-steady states which are the same as (13-16) with space and time averages instead of time averages. Introducing the pressure tensor and assuming that $\int f\, dxdt$ is symmetric about the mean velocities[5] we find next the transition relations are the deHoffman-Teller relations for a gas with $\gamma = \frac{5}{3}$. This means that even for a turbulent flow the standard shock and contact discontinuity relations hold if the fine structure is small enough.[3]

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5. Note we have assumed no dependence on y and z.
6. One might ask where is the corresponding phenomenon in fluid dynamics, but this probably does not occur because the laminar flow ahead of the shock has a higher Reynolds number than the flow behind it.

It is therefore not surprising that the detached shock data computed by Spreiter and Jones (see footnote 1) fit quite well. The major error occurs in computing the temperature, i.e., in one of the higher moments.

The next problem is to examine the fine structure. Only the zero temperature case has been completely investigated and is truly valid. But various pressure tensor hypotheses confirm that qualitatively in most cases it is right. In this case the Vlasov equations reduce to six ordinary differential equations, the equations of motion of ion and electron. We first ask what steady solutions of the one-dimensional equations exist. A major qualitative distinction must be made between situations where there is and where there is not a magnetic field[7-13] in the direction of propagation; case i, $B_1 = 0$ and case ii, $B_1 \neq 0$.

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7. K.W. Morton, J. Fluid Mech. 14, part 3, 369-384 (1962).
8. A. Baños, and A.R. Vernon, Nuovo Cimento XV, 269 (1960).
9. J.H. Adlam and J.E. Allen, Phil. Mag. 3, 448 (1958).
10. K.W. Morton, Phys. Fluids 7, 1800 (1964); "Finite Amplitude Compression Waves in a Collision-Free Plasma", NYO-10434, MF-36, Courant Inst. of Math. Sci., New York Univ. (1964).
11. P.G. Saffman, J. Fluid Mech. 11, 552 (1961).
12. R. Peyret, C.R. Acad, Sc. Paris, t.258 (16 mars 1964) Groupe 2, 2973-2976.
13. A.D. Pataraya, Zh. T.F. 32, 5 (1962) (Soviet Phys., Tech. Phys. 7, 5 (1962).

Two possibilities occur. Either there exists a solitary wave and a family of periodic solutions or there exist only a constant state. For a fixed angle between the magnetic field and the direction of propagation the solitary wave exists if the speed of propagation is small enough.

If there exists a solitary wave and if the equations of motion are modified to include an arbitrarily small amount of friction Morawetz[14] for case (i) and Morton (see reference 10) for case (ii) have shown that there exists a steady state shock structure; i.e. there is a solution leading from one constant state to another. This structure has the following properties:

(i) If $B_1 = 0$, there is a front propagating with the usual shock speed followed by an infinite train of oscillations in all quantities with a characteristic wave length equal to the geometric mean of the two phase lengths.[15] The damping of the oscillations tends to zero as friction vanishes.

(ii) If $B_1 \neq 0$, there is a front propagating with the fast shock speed and preceded by precursors oscillations in the field orientation only. Two characteristic wave lengths occur, the ion and electron phase lengths. In the limit of zero mass ratio only the ion phase length occurs.

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14. C.S. Morawetz, "Magneto-hydrodynamic Shock Structure Using Friction", NYO-8677, Inst. of Math. Sci., New York Univ. (1959).

15. The distance a particle travels along a magnetic line during a complete change phase.

If, on the other hand, the solitary wave does not exist then neither does a shock structure exist. Morton (see also reference 16), has computed the solution of a time-dependent problem. An imposed magnetic field, see Figure 2, drives a plasma ahead of a vacuum field. If the imposed magnetic field is not too large there is a continuous solution which at a fixed large time looks like the corresponding friction solution described above.

If the imposed field is large then there does not exist a continuous solution. Morton admits an inner shock connecting two smooth states but its structure is unknown. Auer et al.[16] find a turbulent flow for the case $B_1 = 0$ but it is not clear that any actual turbulence will have this particular fine structure since there would surely be velocity components in the y,z directions for most particles.

Even in the case of existence, turbulence may be caused by instability. If $B_1 = 0$, the charge separation field E is large for small mass ratio. The electrons drift rapidly across the ions and, as was pointed out by Kellog,[17] the whole flow probably exhibits two-stream instability.

If $B_1 \neq 0$ we might also expect instability in the front. The precursor orientation oscillation may destroy the symmetries in the pressure tensor and Lüst's conditions for stability[18] be

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16. P.L. Auer, H. Hurwitz, Jr., and R.W. Kilb, Phys. Fluids 5, 3 (1962).

17. P.J. Kellog, Phys. Fluids 7, 1555-1571 (1964).

18. R. Lüst, "On the Stability of a Homogeneous Plasma with Non-Isotropic Pressure", in U.S. Atomic Energy Report TID-7582, 154-157 (1959).

violated.

If we examine the data (see reference 2) which are available for the IMP satellite we see that generally the shock will have $B_1 \neq 0$ and a study of the field shows that some kind of precursor ahead of the shock edge occurs. In fact, the state ahead is roughly constant but for an oscillation in the orientation. The data is unfortunately not such that we can find the characteristic wave length or even tell whether it exists at all.

As for the problem of heating a plasma, we can only say that turbulent shocks should probably be expected. The desired temperatures may not be realized because much energy is in the turbulence. But nevertheless the detached shock due to a magnetic singularity[19] (for the earth, a dipole) suggests a mechanism for ionizing and heating a plasma which would then be confined partly by a concave stable front with a vacuum, partly by a shock front, and otherwise by the unionized gas streaming by.

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19. J. Hurley, Phys. Fluids 4, 109 and 854 (1961).

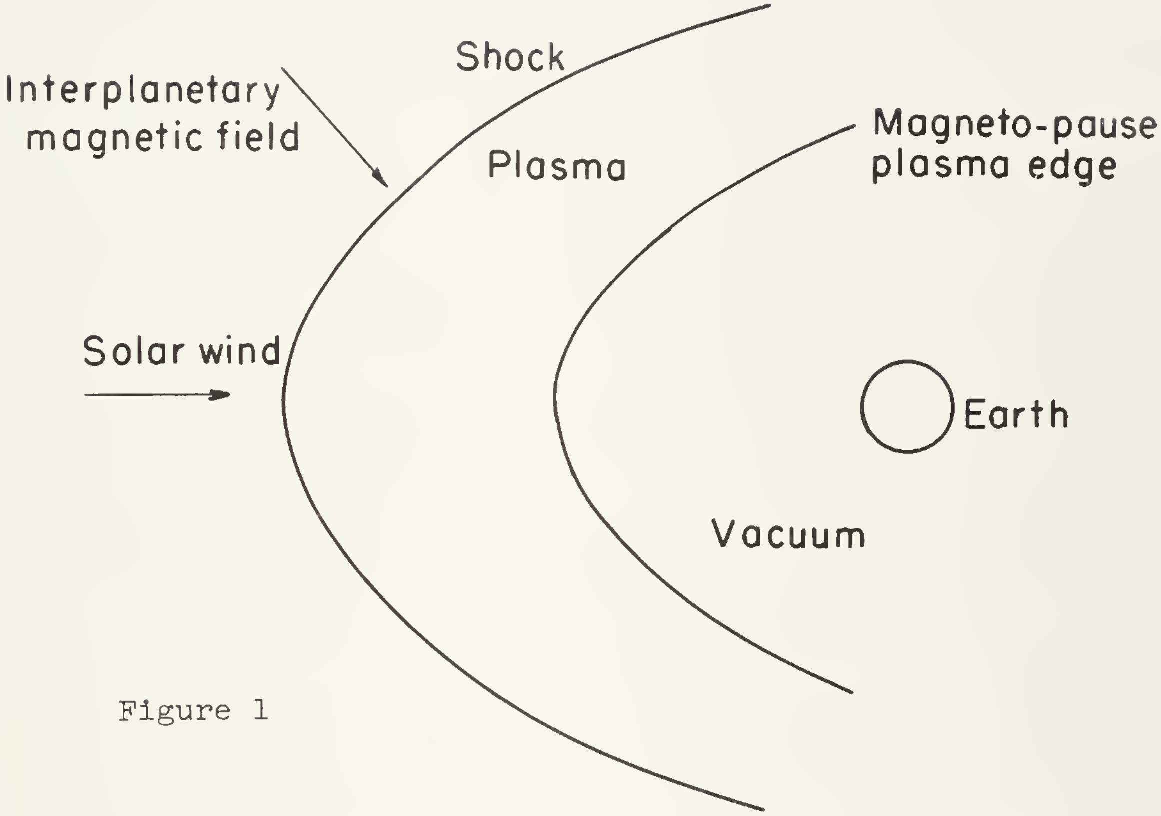

Figure 1

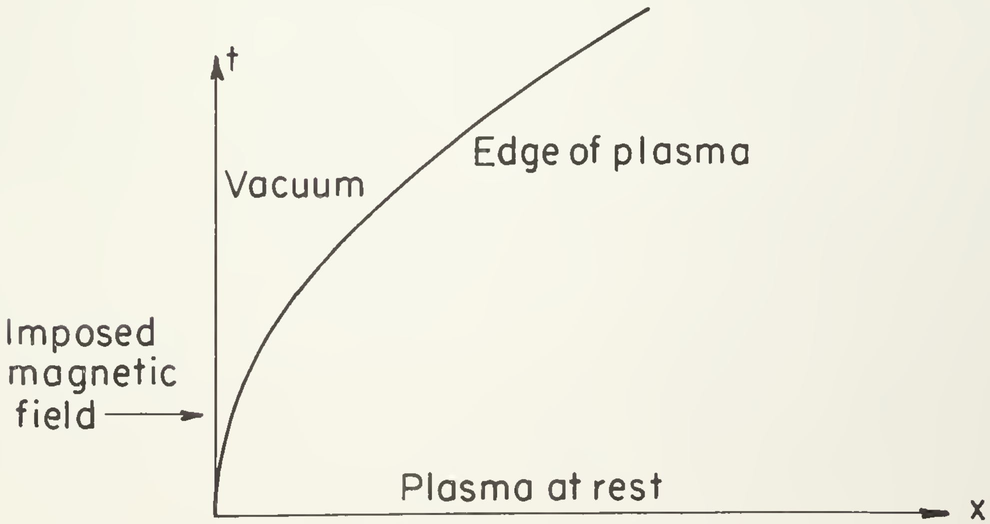

Figure 2